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On the Propagation of Soliton Solution in Comb-like Model

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Abstract:

In present work, comb-like model (toy model has proven valuable to explain and quantify the transport along spin dendrites like for instance nerve cell conduction) as a fractal background medium has been used to derive the well-known nonlinear fractional KdV equation where time evolution operator admits half-order ($\alpha=1/2$). We investigate the effect of the presence of infinite fingers of our suggest model to the propagation of soliton solution along back bone of structure by employing the travelling wave transform method in order to obtain the corresponding soliton solution. The time fractional operator causes a remarkable change on the soliton profile in both width and amplitude. The main results of this study show the sensitive dependence of soliton profile (width and amplitude) on the fractional exponent of time evolution operator. This means that fractal geometry like spines dendrites structure enhanced the propagation of soliton profile along the backbone of the structure due to the increasing of the amplitude and decreasing its width. In addition to there are no effect on the form of soliton with variation of time fractional operator. Finally, we can say that, Comb-like model provide a good geometrical explanation of anomalous transport.

Keywords: Comb-like model; Caputo fractional operator; time fractional, KdV equation.

1. Introduction

In recent years, many number of books and comprehensive reviews have discussed anomalous diffusion process that seem to be present in many fields of sciences i.e. physics, biology, and chemistry, see for instance [1-4] and references therein. As is well known, the phenomena of anomalous or complex transport characterize by the nonlinear growth of the mean-square displacement i.e. $\langle \delta r^2(t) \rangle \approx t^\alpha$, with an exponent $0 < \alpha < 1$. For $\alpha = 1$, this is the hall-mark of normal diffusion process. This type of anomalous behavior is referred to as sub-diffusive processes. Sub-diffusive was observed in many physical and biological phenomena like for example, among other, the motion of traces in living biological cells [5], diffusion in porous or fractal media [6].

Of particular current interest in this is the derivation of some integrable nonlinear partial differential equations (NPDE) like for instance nonlinear kdV equation using a fractal median like Comb-like model as a background medium. In fact, this toy model has proven valuable to explain and quantify the transport along spin dendrites like for instance nerve cell conduction. In other words, Comb-like model provide a good geometrical explanation of anomalous transport.

Noticeably, the Comb-like model was put forward to explain anomalous diffusion in percolation clusters [7-12]. As shown in Fig(1) Comb-like structure consisting of a main backbone located along the x -axis and distributed branches called fingers directed along y -axis, that extend to infinity. This suggest structure pick up the main features of fractal geometry regarding the topological heterogeneity and represents a good example of multiple trapping medium.

Thus, while we try to display a neatly derivation of nonlinear fractional Korteweg de Vries equation (KdVE) using Comb-like model, one could find in principle a direct relation between the geometry of the dendrite spines and anomalous transport equation with fractional derivative operators[13,14].

As a matter of fact, most NPDE that have been used to describe the nonlinear physical phenomena must be solved numerically as a general analytical solution do not exist. However, there are a few NPDE, like for instance the well-known KdV, which in principle one can obtain a neatly analytical solution. These class of nonlinear PDE are completely integrable. Remarkably, the KdV equation is developed to apply in many areas of physics, especially in case that waves can propagate in a weakly nonlinear and dispersion medium. The soliton solution that obtained from KdV is a stable solitary wave, which is a spatially localized pulse propagates at constant velocity.

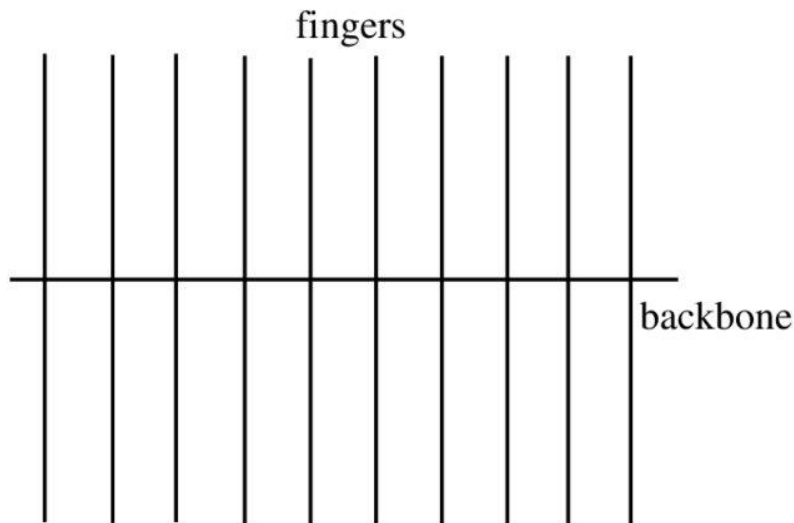
The existence of this spatially permanent soliton solution that obtained from KdV equation results due to the balance between the following two opposite effects, nonlinearity and dispersion. In essence, dispersion tends to spread the nonlinear waves out, while nonlinear effect comes to localize it. Therefor soliton solution propagates under two counter balance effects and has a permanent profile.

Recently, in last decade KdV equation with fractional time and space derivatives has been developed to describe the propagation of nonlinear waves in plasma media [15-17]. In order to explain theoretically these anomalous dynamics, PDE with fractional time and space derivatives have been extensively employed.

This work in principle has two-fold, the first purpose of this work is to give a neatly derivation to the well know nonlinear fractional KdV equation using Comb-like model as a back ground fractal geometry. Concerning section two, we investigate in some details the effect of the presence of infinite fingers of our suggest model to the propagation of soliton solution along back bone of structure. This paper is organizing as follows: The Comb-like model is used to derive the time fractional KdV equation in section 2. Section 3 devoted to solve the obtained time fractional KdV equation. Some notes and remarks are presented in section 4.

2. Time fractional KdV equation:

As is well known, the Comb-like model consists of conducting axis which the main physical phenomena occurs connected with infinite fingers as shown in Fig (1).



Fig(1): Comb-like model

The main feature of our considered model is that the dominant physical event like for instance diffusion, convention or even reaction takes place only along the axis of structure (at $y=0$)[13,14]. Therefore, let us assume that the current density along the backbone of the structure is given by nonlinear function as

$$J_x(x, y, t) = -\delta(y)f(u) \quad (1)$$

Were

$$f(u) = \gamma u^2(x, y, t) - \beta u_{xx}(x, y, t) \quad (2)$$

While the y-component of the current density along the infinite fingers is given by,

$$J_y(x, y, t) = -D \frac{\partial u(x, y, t)}{\partial y} \quad (3)$$

In the above expression, D is the diffusion coefficient along the fingers and assumed to be constant.

By using the continuity equation

$$\frac{\partial u(x, y, t)}{\partial t} = -\nabla \cdot J(x, y, t) = -\left[\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} \right] \quad (4)$$

Inserting equations (1) and (2) into equation (4), one obtains

$$\left[\frac{\partial}{\partial t} - D \frac{\partial^2}{\partial y^2} \right] u(x, y, t) = \delta(y) \frac{\partial}{\partial x} [\gamma u^2(x, y, t) - \beta u_{xx}(x, y, t)] \quad (5)$$

The Green function $G(y, t)$ associated with the homogeneous part of above equation (5) subject to the initial condition $u(x, y, 0) = \delta(x)\delta(y)$ has the following form

$$G(y, t) = \frac{1}{\sqrt{4\pi Dt}} \exp(-y^2/4Dt) \quad (6)$$

Remarkably, the above solution is the well-known Gaussian distribution. Thus, the general solution concerning Eq. (5) may be carried out by integration over the source term for different values of the time parameter.

$$\begin{aligned} u(x, 0, t) &= \int_0^t dt' \int dy' G(y', t-t') \delta(y') \frac{\partial}{\partial x} [\gamma u^2(x, y', t') - \beta u_{xx}(x, y', t')] \\ &= \frac{1}{\sqrt{4\pi D}} \frac{\partial}{\partial x} \int_0^t \frac{1}{(t-t')^{1/2}} [\gamma u^2(x, t') - \beta u_{xx}(x, t')] dt' \\ u(x, t) &= \frac{\Gamma(1/2)}{2\sqrt{\pi} D^{1/2}} {}_0D_t^{-1/2} \frac{\partial}{\partial x} [\gamma u^2(x, t) - \beta u_{xx}(x, t)] \\ u(x, t) &= {}_0D_t^{-1/2} [\gamma u(x, t) u_x(x, t) - \beta' u_{xxx}(x, t)] \end{aligned} \quad (7)$$

Where ${}_0D_t^{-1/2}$ is the Liouville-Riemann fractional integral operator, that is defined by following expression,

$${}_0D_t^{-1/2} f(t) = \frac{1}{\Gamma(1/2)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1/2}} d\tau$$

Operating by $\frac{\partial}{\partial t}$ on both sides of equation (7), one can get

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1/2} [\gamma u(x,t)u_x(x,t) - \beta u_{xxx}(x,t)]$$

It is easy to put above equation in an equivalent form in terms of Caputo differential operator as,

$${}_0^CD_t^{1/2}u(x,t) = -[\gamma u(x,t)u_x(x,t) + \beta u_{xxx}(x,t)] \quad (8)$$

Where ${}_0^CD_t^{1/2}$ is the Caputo derivative operator which defined as,

$${}_0^CD_t^{1/2}f(t) = \frac{1}{\Gamma(1/2)} \int_0^t \frac{df(\tau)/d\tau}{(t-\tau)^{1/2}} d\tau$$

Equation (8) is known as time-fractional KdV equation. In essence, in our toy model, one finds three different events take place inside this structure. Along backbone of structure, solitary wave propagation due to the balance between nonlinear term and dispersion effects. On the other hand particles could be trapped into the fingers with mean square displacement $\langle y^2(t) \rangle \approx Dt$, that admits Gaussian normal diffusion (6) along the fingers.

3. The solution of Time fractional KdV equation

In order to solve the TFKdV equation (8) with $\alpha = 1/2$, where α is the time fractional operator, introducing the following transformation

$$\eta = x - \frac{(vt)^\alpha}{\Gamma(\alpha+1)}, \quad (9)$$

where v is considered as an adjustment parameter in the power law transformation. By using the following properties of fractional derivative,

$${}_x^CD_x^\alpha x^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} x^{(\nu-\alpha)}, \quad (10a)$$

$${}_x^JD_x^\alpha f(u(x)) = {}_x^JD_x^\alpha u(x) \left(\frac{df}{du} \right) \quad (10b)$$

Leads to

$$-v^\alpha \frac{d\Phi(\eta)}{d\eta} + A\Phi(\eta) \frac{d\Phi(\eta)}{d\eta} + B \frac{d^3\Phi(\eta)}{d\eta^3} = 0 \quad (11)$$

This equation is the reduced KdV equation, where $\Phi(\eta) = u(x,t)$ which admits the following soliton solution,

$$\Phi(\xi) = \phi_0 \operatorname{sech}^2 \left[\frac{\eta}{\Delta} \right] \quad (12)$$

with $\phi_0 = \frac{3\nu^\alpha}{A}$, $\Delta = \sqrt{\frac{4B}{\nu^\alpha}}$ are the amplitude and width of the envelope soliton, respectively.

Then the solution of the time- fractional KdV equation (8) takes the following form,

$$u(x,t) = \phi_0 \operatorname{sech}^2 \left[\Delta^{-1} \left(x - \frac{(vt)^\alpha}{\Gamma(\alpha+1)} \right) \right] \quad (13)$$

The different effects of the fractional parameter α on the propagation of envelope wave will investigate in next section.

4. Results and discussion

The main remarks that obtained in the present work can be summarized as follows:

Comb-like model give us an open window about the direct relation between anomalous transport processes and the fractional dynamics. In principle, it can provide us an accessible and effective neatly technique to derive many nonlinear PDE with fractional differential operators. As we have seen, KdV equation with fractional time derivatives has been obtained to describe the propagation of soliton profile along the backbone of the structure.

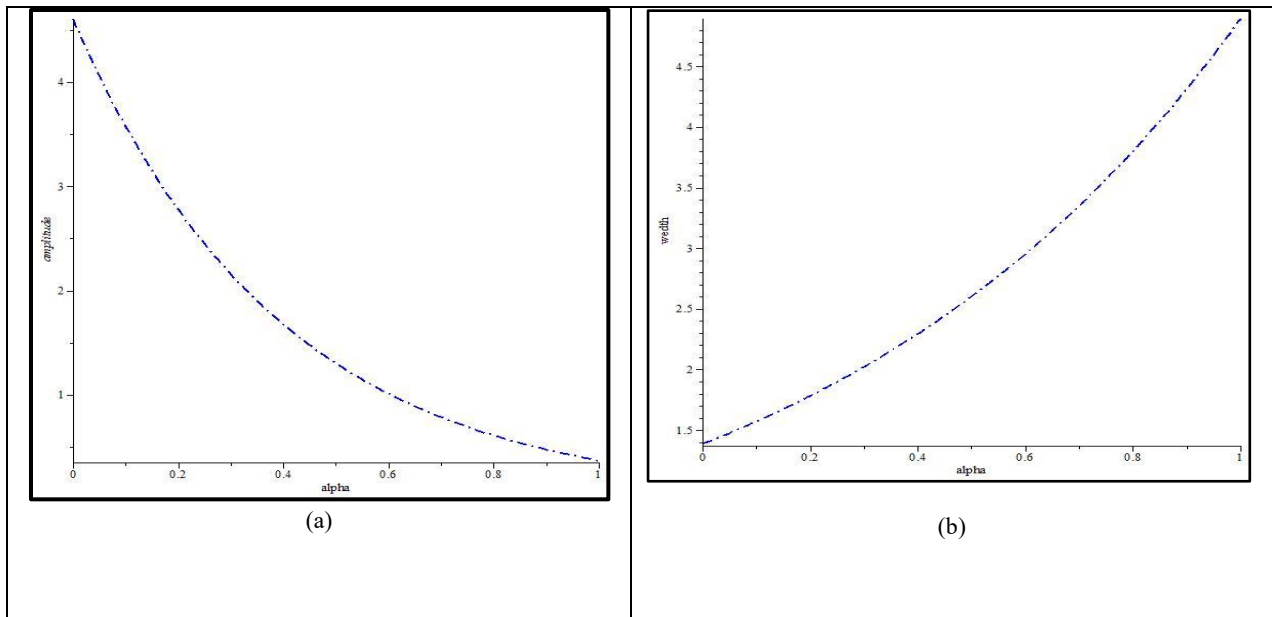
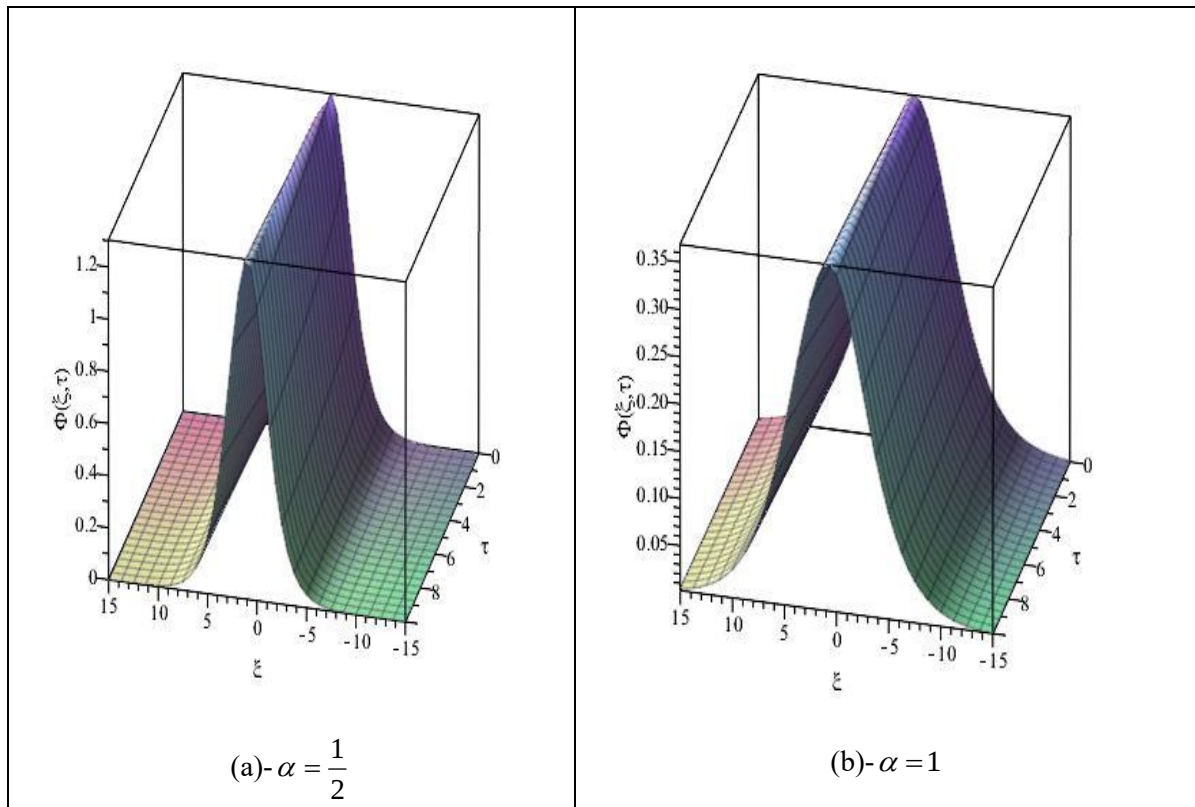
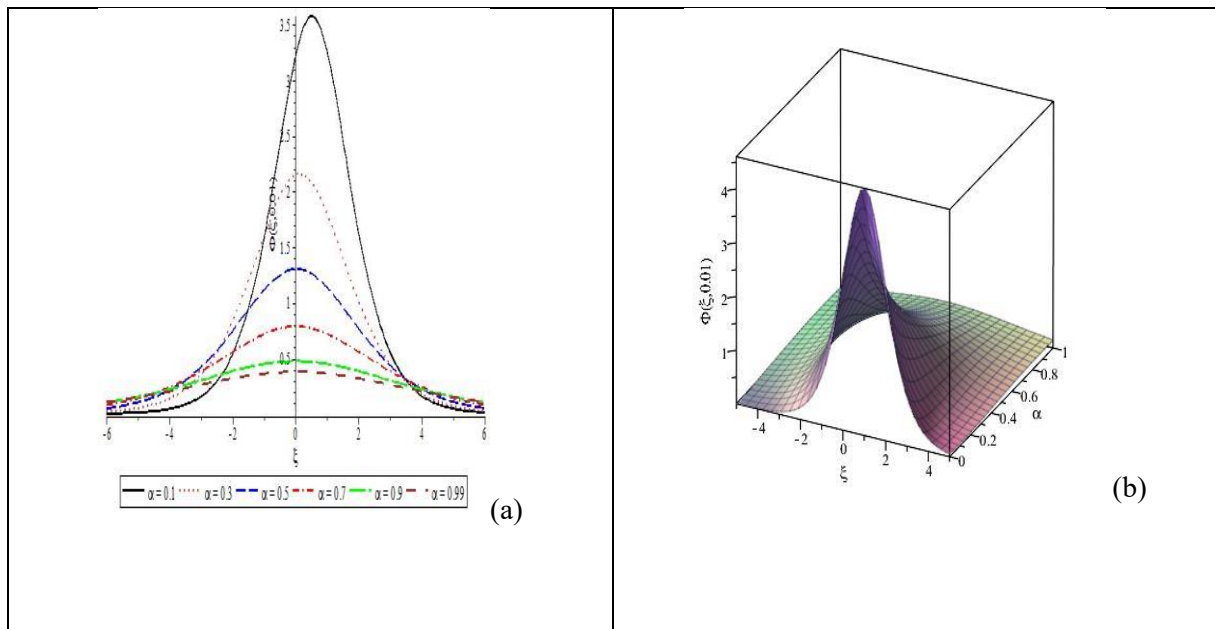


Fig (2): The effect of fractional parameter α on both a-the amplitude and b- the width of envelope wave at $A=0.65$, $B=0.48$, and $\nu = 0.08$

Interestingly, this time fractional operator causes a remarkable change on the soliton profile in both width and amplitude as depicted in Fig. (2), where the amplitude decreased with increasing α , on the contrary width increased with increasing α . This means that fractal geometry like spines dendrites structure enhanced the propagation of soliton profile along the backbone of the structure due to the increasing of the amplitude and decreasing its width.



Fig(3): 3dimension envelope wave at a- $\alpha = \frac{1}{2}$ and b- $\alpha = 1$, at $A=0.65$, $B=0.48$,



Fig(4): The dependence of envelope wave on the fractional parameter α , at $A=0.65$, $B=0.48$, and $\nu = 0.08$.

On other hand, there are no effect on the form of soliton with variation of time fractional operator, but the variation on the width, amplitude, and the position of the peak of envelope soliton as shown in both 3 and 2-dimensional graphs (3) and (4) respectively.

In essence, one can safely say that, although the model presented here is a toy-model rather than model of real physical situation, but indeed gives us an obvious and vital idea concerning systems exhibits multiple trapping.

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